Extended Lyapunov Stability Criterion Using a Nonlinear Algebraic Relation with Application to Adaptive Control

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A new nonlinear stability criterion is developed by use of a class of Lyapunov functionals for model-reference adaptive systems (MRAS). Results are compared with traditional results, and a comparative design technique is used to illustrate its function in improving the transient response of an MRAS controller. For a particular system structure and class of input signals, the new stability criterion is shown to include traditional sufficiency stability conditions as a special case. An example is cited to illustrate the use of the nonlinear criterion and its definite advantages in helping improve the adaptive error transient response of a system. Analysis of results is effected by use of a linearization technique on the resulting adaptive equations.

I. Introduction

THE area of adaptive control theory has been under considerable investigation during the last decade, and in particular that of model-reference adaptive control, as evidenced by a plethora of papers on the subject.[1-28] In most cases, sufficient conditions insuring asymptotic stability of the system error or, in some cases, practical stability have been presented. In almost no instances, however, is the rate of convergence considered in any detail. This is in part because of the seemingly inherent complexity of the mathematical expressions involved, especially when Lyapunov theory is used.[29,30] When practical application is involved, however, practical stability[31] and practical convergence become of great importance. Carried to the extreme it is conceivable that a fast converging adaptive technique that does not insure stability would be a much more likely candidate than a technique that insures stability, but whose convergence rate was extremely slow.

This paper presents a new technique for obtaining a stability criterion for a class of model-reference adaptive systems (MRAS), using Lyapunov theory. The method is exact, and it results in a new, nonlinear matrix relationship.[32] Demonstration of the usefulness of the matrix equation is given in extending stability bounds beyond those of a classical linear criterion, and an algorithm for solving the resulting nonlinear relationship is presented.

II. Problem Statement

The class of problems to be considered is of the type in Refs. 5, 8, 18, and 21. Given a basic plant of the form

$$\dot{x}_p(t) = A_p x(t) + B_p U(t) \tag{1}$$

it is desired to force the system in Eq. (1) to track a linear time-invariant model of the form

$$\dot{x}_m(t) = A_m x_m(t) + B_m U(t) \tag{2}$$

by forcing the system error e

$$e(t) = x_m(t) - x_p(t) \tag{3}$$

to eventually approach zero, where $x_m(t)$, $x_p(t)$, $e(t)$ are n vectors; A_p, A_m are $n \times n$ matrices; B_p, B_m are $n \times r$ matrices; and U is an r vector. Differentiating Eq. (3) with respect to time, and substituting Eqs. (1) and (2), it can be shown that

$$\dot{e}(t) = A_m e + A x_p(t) + B\ U(t) \tag{4}$$

$$A = [a_{ij}^m - a_{ij}^p] \tag{5}$$

$$B = [b_{ij}^m - b_{ij}^p] \tag{6}$$

A_p, B_p are of the form

$$A_p = [c_{ij}^a + K_{ij}^a(t)] \tag{7}$$

$$B_p = [c_{ij}^b + K_{ij}^b(t)] \tag{8}$$

where c_{ij}^a, c_{ij}^b represent constant, unknown plant parameters, and $K_{ij}^a(t)$, $K_{ij}^b(t)$ represent adaptive gains, which insure that $e(t)$ approaches zero asymptotically. The form in Eqs. (7) and (8) assumes the ability to adjust the system parameters directly. Differentiating Eqs. (7) and (8) results in

$$\dot{a}_{ij}^p = \dot{K}_{ij}^a(t) \tag{9}$$

$$\dot{b}_{ij}^p = \dot{K}_{ij}^b(t) \tag{10}$$

Two different adaptive systems are considered, with Lyapunov functions of the form[8]

$$V = e^T Q e + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_{ij}} \left[a_{ij} + \beta_{ij} \sum_{k=1}^{n} e_k q_{ki} x_{pj} \right]^2 + \sum_{i=1}^{n} \sum_{j=1}^{r} \frac{1}{\gamma_{ij}} \left[b_{ij} + \delta_{ij} \sum_{k=1}^{n} e_k q_{ki} U_j \right]^2 \tag{11}$$

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and corresponding adaptive gains

$$K_{ij}^{a}(t)=\alpha_{ij}\int_{t_0}^{t} S_{ij}\mathrm{d}t+\beta_{ij}S_{ij}$$

$$K_{ij}^{b}(t)=\gamma_{ij}\int_{t_0}^{t} Y_{ij}\mathrm{d}t+\delta_{ij}Y_{ij} \tag{12}$$

as well as[21]

$$\begin{aligned}V=e^TQe+&\sum_{i,j=1}^{n}\frac{1}{\alpha_{ij}}\left\{a_{ij}+\beta_{ij}\sum_{k=1}^{n}e_kq_{ki}x_{pj}\right.\\&\left.+\rho_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{k=1}^{n}e_kq_{ki}x_{pj}\right]\right\}^2\\&+\sum_{i,j=1}^{n}\rho_{ij}\left[\sum_{k=1}^{n}e_kq_{ki}x_{pj}\right]^2\\&+\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{1}{\gamma_{ij}}\left\{b_{ij}+\delta_{ij}\sum_{k=1}^{n}e_kq_{ki}U_j\right.\\&\left.+\sigma_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{k=1}^{n}e_kq_{ki}U_j\right]\right\}^2\\&+\sum_{i=1}^{n}\sum_{j=1}^{r}\sigma_{ij}\left[\sum_{k=1}^{n}e_kq_{ki}U_j\right]^2\end{aligned} \tag{13}$$

with corresponding adaptive gains

$$K_{ij}^{a}(t)=\alpha_{ij}\int_{t_0}^{t} S_{ij}\mathrm{d}t+\beta_{ij}S_{ij}+\rho_{ij}\frac{\mathrm{d}}{\mathrm{d}t}S_{ij}$$

$$K_{ij}^{b}(t)=\gamma_{ij}\int_{t_0}^{t} Y_{ij}\mathrm{d}t+\delta_{ij}Y_{ij}+\sigma_{ij}\frac{\mathrm{d}}{\mathrm{d}t}Y_{ij} \tag{14}$$

where

$$S_{ij}=\sum_{k=1}^{n}e_kq_{ki}x_{pj}$$

$$Y_{ij}=\sum_{k=1}^{n}e_kq_{ki}U_j \tag{15}$$

$\alpha_{ij},\beta_{ij},\rho_{ij},\gamma_{ij},\delta_{ij},\sigma_{ij}$ are positive constants, and the q_{ki} are elements of a positive-definite (p.d.) symmetric Q matrix to be determined. Performing the operation

$$\dot{V}=\nabla V\cdot\dot{e}(t)$$

$$\nabla V=\left[\frac{\partial V}{\partial e_1}\;\frac{\partial V}{\partial e_2}\;\frac{\partial V}{\partial e_3}\;\cdots\right] \tag{16}$$

on Eqs. (11) and (13), and substituting Eqs. (12) and (14), respectively, yields the same result, namely,

$$\begin{aligned}\dot{V}=&\overset{①}{e^T(A_m{}^TQ+QA_m)e}-2\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}\\&\times\overset{②}{\left[\sum_{k=1}^{n}e_kq_{ki}x_{pj}\right]^2}\\&-2\sum_{i=1}^{n}\sum_{j=1}^{r}\delta_{ij}\overset{③}{\left[\sum_{k=1}^{n}e_kq_{ki}U_j\right]^2}\end{aligned} \tag{17}$$

The last two terms in Eq. (17) are negative definite. It is well-known[7] that, if A_m is a stable matrix, there exists a p.d., symmetric Q matrix satisfying

$$A_m{}^TQ+QA_m=-C \qquad C \text{ p.d., symmetric} \tag{18}$$

Hence, from Lyapunov theory,[31,33] $\boldsymbol{e}(t)$ goes to zero and Eq. (1) is asymptotically stable in $\boldsymbol{e}$. Use of Eq. (18), however, implies a sufficiency condition for stability, the information contained in the second and third terms of Eq. (17) being disregarded. Because of the mathematical complexity inherent with Lyapunov theory, in practice, sufficiency conditions are treated as necessary conditions; i.e., a Q matrix not satisfying Eq. (18) would not be used, even though Eq. (17) might be n.d. In the next section, the derivation of a sufficiency condition analogous to Eq. (18) will be given. Then a following section will discuss the meaning of the new criterion in terms of Eq. (18) and the adaptive error convergence rate.

III. Derivation of the Nonlinear Algebraic Stability Criterion

Although Eq. (18), coupled with Eq. (17), will insure an asymptotically stable system, the question arises as to whether or not the additional information available from the second and third terms might be used to obtain a relationship slightly different from Eq. (18), and whether this new criterion allows a greater selection of Q matrices than is available using Eq. (18).

For the derivation that follows, there are four conditions that must occur: 1) A_m is the phase-variable form; 2) at least one nonzero external input is present; 3) B_m,B_p contain all zeroes except for the last row; and 4) there is at least one entry of B_p which is adapted.

Under these conditions, the third term of the $\dot{V}$ function in Eq. (17) may be expanded as follows:

$$\begin{aligned}③=&2\sum_{i=1}^{n}\sum_{j=1}^{r}\delta_{ij}\left[\sum_{k=1}^{n}e_kq_{ki}U_j\right]^2\\=&2\left\{\delta_{n1}[(e_1q_{1n}+e_2q_{2n}+\ldots+e_nq_{nn})U_1]^2\right.\\&+\delta_{n2}[(e_1q_{1n}+e_2q_{2n}+\ldots+e_nq_{nn})U_2]^2+\ldots\\&\left.+\delta_{nr}[(e_1q_{1n}+e_2q_{2n}+\ldots+e_nq_{nn}]U_r]^2\right\}\end{aligned} \tag{19}$$

which reduces to

$$\begin{aligned}③=&2(\delta_{n1}U_1{}^2+\delta_{n2}U_2{}^2+\ldots+\delta_{nr}U_r{}^2)\\&\times[e_1q_{1n}+e_2q_{2n}+\ldots+e_nq_{nn}]^2\end{aligned} \tag{20}$$

The squared factor in Eq. (20) is expanded as follows

$$\begin{aligned}&(e_1q_{1n}+e_2q_{2n}+\ldots+e_nq_{nn})(e_1q_{1n}+e_2q_{2n}+\ldots\\&+e_nq_{nn})=e_1{}^2q_{1n}{}^2+2e_1e_2q_{1n}q_{2n}\\&+2e_1e_3q_{1n}q_{3n}+\ldots+2e_1e_nq_{1n}q_{nn}+e_2{}^2q_{2n}{}^2\\&+2e_2e_3q_{2n}q_{3n}+2e_2e_4q_{2n}q_{4n}+\ldots\\&+2e_2e_nq_{2n}q_{nn}+\ldots e_n{}^2q_{nn}{}^2\end{aligned} \tag{21}$$

which may be put in matrix form as

$$e^T\begin{bmatrix}q_{1n}^2 & q_{1n}q_{2n} & q_{1n}q_{3n} & \cdots & q_{1n}q_{nn}\\ q_{2n}q_{1n} & q_{2n}^2 & q_{2n}q_{3n} & \cdots & q_{2n}q_{nn}\\ \vdots & & & & \\ q_{nn}q_{1n} & q_{nn}q_{2n} & q_{nn}q_{3n} & \cdots & q_{nn}^2\end{bmatrix}e \tag{22}$$

Define Ω as

$$\Omega=2(\delta_{n1}U_1{}^2+\delta_{n2}U_2{}^2+\ldots+\delta_{nr}U_r{}^2) \tag{23}$$

where at least one of the δ_{nj} terms is nonzero. With

$$\hat{q}=\begin{bmatrix} q_{1n} \\ q_{2n} \\ \cdot \\ \cdot \\ \cdot \\ q_{nn} \end{bmatrix} \tag{24}$$

Eq. (20) may be written as

$$\Omega e^T(\hat{q}\hat{q}^T)e \tag{25}$$

By using Eq. (25), Eq. (17) may be rewritten as

$$\dot{V}=e^T\overset{①}{(A_m{}^TQ+QA_m)}e-\overset{②}{\Omega e^T\hat{q}\hat{q}^T e}$$
$$-2\sum_{i=1}^{n}\sum_{j=1}^{n}\Big[\overset{③}{\sum_{k=1}^{n} e_k q_{ki} x_{pj}}\Big]^2\beta_{ij} \tag{26}$$

which simplifies to

$$\dot{V}=e^T We-2\sum_{i=1}^{n}\sum_{j=1}^{n}\Big[\sum_{k=1}^{n} e_k q_{ki} x_{pj}\Big]^2\beta_{ij} \tag{27}$$

where

$$W=A_m{}^TQ+QA_m-\Omega\hat{q}\hat{q}^T \tag{28}$$

If W is n.d., then Eq. (17) is n.d., and if the resulting Q is p.d., then the system of Eqs. (1-3) will be asymptotically stable in e. Under the constraint of the earlier restrictions, Eq. (28) then may be used in place of Eq. (18) for insuring asymptotic stability. Based on the preceding, the following conjecture is advanced.

Conjecture: If A_m is an $n\times n$ real matrix with eigenvalues λ_n and $Re\{\lambda_n\}<0\forall n$, W is a negative-definite symmetric matrix, Ω is a positive real number, and $\hat{q}$ is the nth column of an $n\times n$ real matrix $Q=(q_1q_2...\hat{q}_n)$, then there exists a unique positive-definite symmetric $n\times n$ matrix Q satisfying Eq. (28).

For the present, a simple algorithm for solving Eq. (28) is proposed, as detailed in the Appendix. Based on a form of the Newton-Raphson iteration, it provides a computationally straightforward solution to Eq. (28), starting with an assumed solution that neglects the nonlinear term. Whether the conjecture is true or not is immaterial from the standpoint of the importance of the result given in Eq. (28). All that is required for the original system to be asymptotically stable in e is that 1) Q be p.d., symmetric; and 2) $W=A_m{}^TQ+QA_m-\Omega\hat{q}\hat{q}^T$ be n.d., symmetric. If the conjecture is true, it simply means that it is very simple to obtain solutions to Eq. (28) satisfying requirements 1 and 2. The suggested numerical approach is just a simple way of getting such solutions; conditions 1 and 2 would need to be verified in addition.

Use of Eq. (28) therefore can make it possible to obtain a wider choice of q_{ij} values. The interpretation of the meaning of this larger choice of q_{ij} elements in terms of the adapted system error dynamics is discussed in Sec. IV. The reason that the allowable range of values of q_{ij} is larger is that the new sufficiency condition [Eq. (28)] is used instead of Eq. (18). If only Eq. (18) is used, then the capability of using other information from the Lyapunov $\dot{V}$ function [i.e., terms ② and ③ of Eq. (26)] is ignored.

IV. Relationship Between Q Element Selection and Adaptive Error Transient Response

It has been shown[8,15] that, for a general nth-order adaptive system in the phase variable form, the adaptive error characteristic equation can be approximated by

$$1+\Big\{\Big[\overset{①}{\sum_{i=1}^{n} q_{in}s^{i-1}}\Big]\Big[\overset{②}{\sum_{j=1}^{p} K_j s^{j-1}}\Big]\Big\}\Big/\ [s\Delta_m(s)]=0 \tag{29}$$

where q_{ij} are elements of the p.d. Q matrix; $\Delta_m(s)$ is a model characteristic equation; K_j are lumped constants, which are a function of the adaptive gain parameters α, β, ρ, γ, δ, σ; p is the type of adaptation $p=1$,[5] $p=2$,[8], $p=3$[21]; s is a Laplace operator; and $K_1,K_2,K_3>0$.

Equation (29) was developed by expanding $\dot{e}$ in Eq. (4) and the $K_{ij}{}^a(t)$, $K_{ij}^b(t)$ in Eqs. (12), 14), and (15) in a Taylor series about the operating point

$$e=0$$

$$x_p{}^0=x_m{}^0=[x_{1_m}^0\quad 0\quad 0.......0]^T$$

because

$$x_{2_p}^0=x_{2_m}^0=\dot{x}_{1_p}=0$$

$$x_{3_p}^0=x_{3_m}^0=\ddot{x}_{1_p}=0$$

$$\vdots$$

$$x_{n_p}^0=x_{n_m}^0=x_{1_p}^{(n-1)}=0$$

$U=U^0=$ constant input vector assumed

$$a_{ij}^m=a_{ij}^p$$

and truncating after the linear terms of the expansion. The superscript 0 refers to the corresponding Taylor series expansion point value. From Lyapunov theory, the system is guaranteed to converge to $e=0$. Next it will be shown how Eq. (29) can be used in designing MRAS systems by relating adaptive parameters to the error transient response in a straightforward manner.

Expanding term ① of Eq. (29), one obtains

$$\sum_{i=1}^{n} q_{in}s^{i-1}=q_{1n}+q_{2n}s+q_{3n}s^2+...+q_{nn}s^{n-1} \tag{30}$$

which is a polynomial in elements of Q, specifically elements of the last row and column (since Q is symmetric). It now can be seen that the numerical values of Q enter into the dynamical relationship of e by acting as coefficients of a polynomial in s representing compensating zeros of a root-locus-type expression. The zeros are the solutions to

$$s^{n-1}+\frac{q_{(n-1)n}}{q_{nn}}s^{n-2}+\frac{q_{(n-2)n}}{q_{nn}}s^{n-3}$$
$$+...+\frac{q_{1n}}{q_{nn}}=0 \tag{31}$$

The choice of the q_{ij} elements, coupled with the actual numerical value of "k" in Eq. (29), then helps to determine the closed-loop error poles given by Eq. (29). The K_j terms in term ② of Eq. (29) are of the form

$$K_1=\alpha_{n1}x_{1_m}^{0^2}+(\gamma_{n1})U_1{}^{0^2}$$
$$+(\gamma_{n2})U_2{}^{0^2}+...+(\gamma_{nr})U_r{}^{0^2}\quad \text{see Ref. 5} \tag{32a}$$

$$K_2=\beta_{n1}x_{1_m}^{0^2}+(\delta_{n1})U_1{}^{0^2}+(\delta_{n2})U_2{}^{0^2}$$
$$+...+(\delta_{nr})U_r{}^{0^2}\qquad \text{see Ref 8} \tag{32b}$$

$$K_3=\rho_{n1}x_{1_m}^{0^2}+(\sigma_{n1})U_1{}^{0^2}+(\sigma_{n2})U_2{}^{0^2}$$
$$+(\sigma_{nr})U_r{}^{0^2}\qquad \text{see Ref. 21} \tag{32c}$$

From Eqs. (29) and (32), the convergence rate then is a function of 1) the model matrix A_m; 2) the adaptive gain parameters $\alpha,\beta,\rho,\gamma,\delta,\sigma$; and 3) state and input values.

The "root-locus gain" k from Eq. (29) is a function of the product q_{nn} and K_ℓ, where K_ℓ is either K_1, K_2, or K_3, as given previously. The relationship between the parameter Ω, knowledge of which allows a greater range of roots than if it is assumed to be zero, and the root locus gain k now can be seen. If $K_\ell = K_2$, K_2 from Eq. (32) then can be written as

$$K_2 = \beta_{n1} x_{1_m}^{o2} + \sum_{j=1}^{r} \delta_{nj} U_j^{o2} \tag{33}$$

$$= \delta_{n1} x_{1_m}^{o2} + (\Omega/2) \tag{34}$$

But

$$\beta_{n1} x_{1_m}^{o2} \geq 0 \qquad \forall x_{1_m}(t),\ \beta_{n1} \qquad \text{so}$$

$$K_2 \geq \Omega/2 \tag{35}$$

A necessary condition for Q to be p.d. is that $q_{nn} > 0$; therefore,

$$k = q_{nn} K_2 = q_{nn} [\beta_{n1} x_{1_m}^{o2} + (\Omega/2)] \geq q_{nn} (\Omega/2) \tag{36}$$

It should be clear from Eq. (36), then, that Ω represents a proportion of the root locus gain k. This is analogous to the case in linear systems analysis, wherein a system would be stable for some $k > k_{\min}$, as shown in Fig. 1.

As a simple illustration of the expanded form of Eq. (29), for a second-order system of the type in Ref. 21 ($p = 3$) with one-input and one-output with transfer function of the form

$$G_m(s) = \frac{2}{s^2 + 2s + 2}$$

$$G_p(s) = \frac{c}{s^2 + as + b} \qquad a, b, c \text{ unknown}$$

Eq. (29) becomes

$$1 + \frac{(q_{22} K_3)(s + q_{12/q_{22}})(s^2 + K_2/K_3 s + K_1/K_3)}{s(s^2 + 2s + 2)} = 0$$

$$\left.\begin{aligned} K_1 &= \alpha_{21} x_{1_m}^{o2} + \gamma_{11} U_1^{o2} \\ K_2 &= \beta_{21} x_{1_m}^{o2} + \delta_{11} U_1^{o2} \\ K_3 &= \rho_{21} x_{1_m}^{o2} + \sigma_{11} U_1^{o2} \end{aligned}\right\} \begin{array}{l} \text{const} \\ \alpha,\beta,\rho,\gamma,\delta,\sigma \text{ used in Eqs. (12)} \\ \text{and (14)} \end{array}$$

The stability conditon in Eq. (18) is analogous to the case in Ref. 5, where the resulting $\dot{V}$ function is

$$\dot{V} = e^T (A_m{}^T Q + Q A_m) e \tag{37}$$

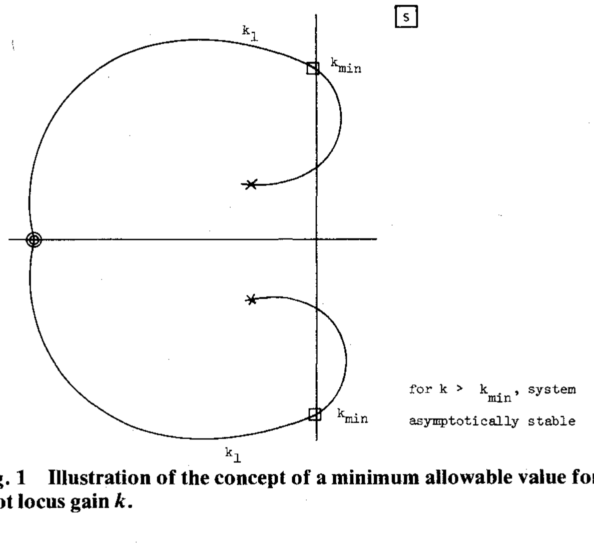

Fig. 1 Illustration of the concept of a minimum allowable value for a root locus gain k.

Comparing Eq. (37) with Eq. (17), it is clear that, using the adaptation method in Ref. 5, the only way to meet the Lyapunov stability conditions is for Eq. (18) to hold; however, information is available such that it is not necessary for Eq. (18) to be true. With Eq. (28), one can use some of that new information available in Eq. (17) which is not available in Eq. (37). Because of Eq. (29), a wider range of values on the zeros of term ① then is available to the designer, offering him the opportunity to get a better transient adaptive error response by (hopefully) placing closed-loop error poles further out in the left-hand s plane than is possible by use of Eq. (18) alone.

The concept is illustrated in Fig. 2 for a second-order system with linearized error characteristic equation

$$1 + [k(s + q_{12/q_2})]/[s \Delta_m(s)] = 0$$

It should be clear from Fig. 2 that it would be desirable to have the zero q_{12}/q_{22} as far out in the left hand plane (LHP) as possible. Use of the K_ℓ terms in Eq. (29) then would supply additional zeros (which may be placed anywhere in the LHP), which may be used to clean up the root locus and keep all loci as far out in the LHP as possible.

In order to use Eq. (28) in a practical manner, some information on a lower bound of Ω in Eq. (23) is required. This is because the $\alpha,\beta,\rho,\gamma,\delta,\sigma$, q_{ij} coefficients must be constant. If one were to use Ω_1 in Eq. (28) for a given U and obtain a root domain D_1 for Eq. (31), and then U changes, a new acceptable domain $D_2 \neq D_1$ results, and new q_{ij} values must be used. In order to eliminate this problem, knowledge of a lower bound on each of the inputs U_i must be available, i.e.,

$$\inf_t \{ |U_i(t)| \} = |U_i|^{\min} \tag{38}$$

This minimum is needed because, from Eq. (28), as Ω increases positively from 0, new q_{ij} bounds are obtained, but they are a function of the input magnitudes. Use of Eq. (38) finds the smallest extended bound on the q_{ij} ratios, which always will guarantee an asymptotically stable system.

In many physical problems, the requirement of a priori knowledge of a lower bound on input magnitudes is not at all restrictive. In chemical processing, a minimum value of input product mix often is required for proper operation. A re-entering space shuttle requires a minimum glide path angle for safe re-entry. Just as with any control process, the more conservative the estimate (of inputs), the more restrictive the stability results are, with a lower stability bound given by the linear criterion result [Eq. (18)].

The need for Eq. (38) is illustrated in Fig. 2 for the case $n = 3$, wherein the roots of Eq. (31) are written as

$$s^2 + q_{23}/q_{33} s + q_{13}/q_{33} = (s + r_1)(s + r_2) \tag{39}$$

and only real roots of Eq. (39) are considered, since, in practice, complex zeroes are avoided whenever possible. Figure 3 is a "root-domain" plot of r_1 vs r_2 for some third-order adaptive system. It is clear that, as Ω increases, the allowable r_1 –

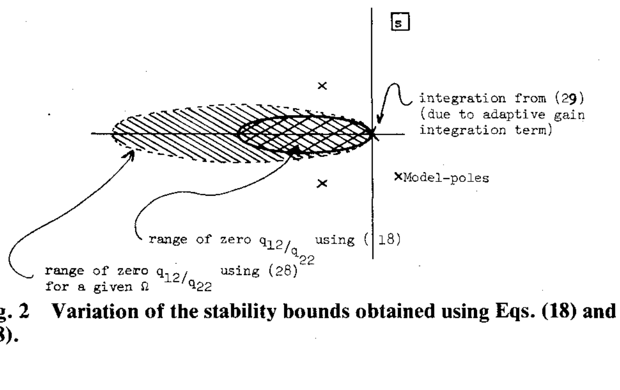

Fig. 2 Variation of the stability bounds obtained using Eqs. (18) and (28).

Ω_2 domain D expands. As $U(t)$ varies, so does Ω, and hence the corresponding domain D. It is intuitive from Eqs. (23) and (28) that, if D_1 is the root domain corresponding to Ω_1 and if D_2 is the corresponding domain for $\Omega_2 > \Omega_1$, then D_1 is a proper subset of $D_2, D_1 \epsilon D_2$.

Those cases in which the domain D is greater than that allowed by Eq. (18) represent solutions of Eq. (28) of the form

$$\left\{\begin{bmatrix}\text{indefinite}\\ \text{matrix}\end{bmatrix}\right\} - \left\{\begin{bmatrix}\text{positive}\\ \text{semi-definite}\\ \text{matrix}\end{bmatrix}\right\} = \left\{\begin{bmatrix}\text{negative-}\\ \text{definite}\\ \text{matrix}\end{bmatrix}\right\}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\{A_m{}^T Q + QA_m\} - \{\Omega \hat{q}\hat{q}^T\} = \{-C\} \tag{40}$$

V. Illustrations of Use of the Extended Stability Criterion

In order to illustrate the practical use of the extended stability bounding criteria derived in Eq. (28), a simple example will be used.

Example: Given the second-order model

$$G_m(s) = 2/(s^2+2s+2) \tag{41}$$

and the plant

$$G_p(s) = b/(s^2+cs+d) \tag{42}$$

determine a stability bound allowed using Eq. (28). Suppose that there is only one input, and

$$\inf_t \{ |U(t)| \} = |U|^{\min} = \sqrt{5}$$

By use of Eq. (29), the error characteristic equation becomes

$$1 + [K(s+a)]/[s(s^2+2s+2)] = 0 \tag{43}$$

not considering term ② of Eq. (29) at this time. It is clear that it is the range of values on $a = q_{12}/q_{22}$ that is of interest. It can be shown[22] that the range of values of a to insure stability for all positive k by use of the traditional criteria in Eq. (18) is

$$0 < a < 2 \tag{44}$$

Using $\delta_1 = 1$, $\Omega_{\min}$ from Eq. (23) becomes

$$\Omega_{\min} = 2\,\delta_1(|U|^{\min})^2 = 2(1)\ (5) = 10 \tag{45}$$

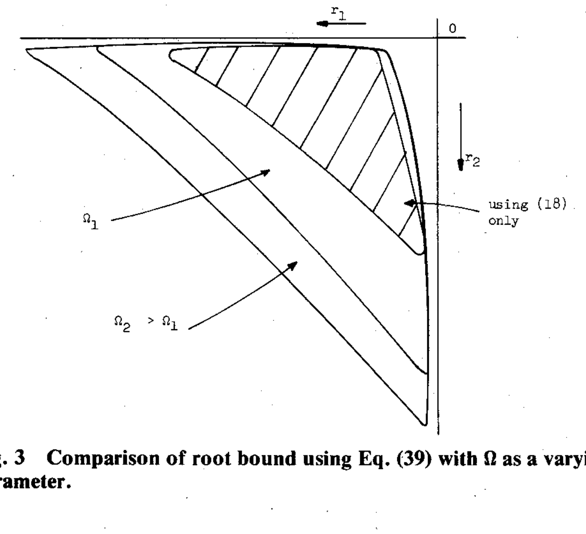

Fig. 3 Comparison of root bound using Eq. (39) with Ω as a varying parameter.

From Eq. (41),

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

and Eq. (28) becomes

$$\overset{①}{\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}}$$

$$-\Omega \overset{②}{\begin{bmatrix} q_{12}^2 & q_{12}q_{22} \\ q_{12}q_{22} & q_{22}^2 \end{bmatrix}} = W \tag{46}$$

where W is required to be negative definite. By use of

$$Q = \begin{bmatrix} 50 & 6 \\ 6 & 1 \end{bmatrix} \qquad q_{12}/q_{22} = 6 \tag{47}$$

Eq. (46) becomes

$$\begin{bmatrix} -24 & 36 \\ 36 & 8 \end{bmatrix} - 10\begin{bmatrix} 36 & 6 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} -384 & -24 \\ -24 & -2 \end{bmatrix} \tag{48}$$

It is clear from Eq. (48) that the linear relationship [Eq. (18)] traditionally used yields an indefinite relationship as shown in term ① of Eq. (48), but the nonlinear criterion [Eq. (28)] yields an n.d. solution as shown in Eq. (48).

Comparing the results in Eq. (44) for the linear criterion, and using Eq. (47) with the nonlinear criterion, it is clear the zero compensator a may be placed three times as far out in the LHP by use of Eq. (28). It is easy to show oneself that a can vary continuously from 0 to 6^+ in fact. The two bounds are shown in Fig. 4.

Let us now actually design an adaptive compensator for this example. Using the adaptive method in Ref. 8, and using Eq. (29), one additional zero compensator, represented by K_1/K_2 may be placed anywhere in the LHP. The root locus equation from Eq. (29) becomes

$$1 + [q_{22}K_2(s + \overline{q_{12}/q_{22}}^{\,a})(s + \overline{K_1/K_2}^{\,b})]/[s(s^2+2s+2)]$$

By use of the linear criterion [Eq. (18)], and Eq. (44), $a = 2 - \epsilon$ was selected, where ϵ is a very small positive number. But, using the nonlinear criterion [Eqs. (28) and (47)], $a = 6$ was chosen. For comparative purposes, $b = 6$ was selected both for the linear and nonlinear criterion cases. The root loci for the two cases are shown in Figs. 5 and 6.

By use of the adaptive gain parameters listed in Table 1, the example was simulated and the results were shown in Fig. 7. Only the two terms b and d in Eq. (42) were adapted, c being assumed equal to the model value (=2), and the adapted parameters initialized at $b=2$ and $d=1.9$. As expected, the error for the nonlinear criterion case decayed at a rate about 2½ times that of the linear criterion case, because

$$\tau_{\substack{\text{dominant}\\ \text{linear}}} \simeq 0.5 \text{ sec} \qquad \tau_{\substack{\text{dominant}\\ \text{nonlinear}}} \simeq 0.2 \text{ sec}$$

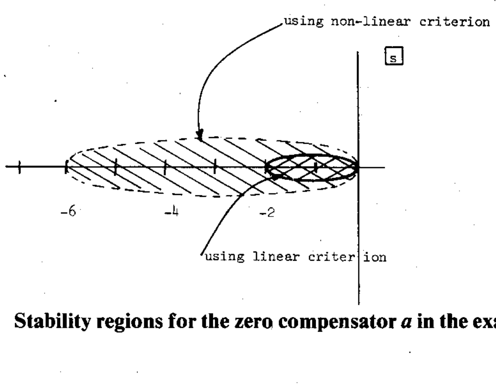

Fig. 4 Stability regions for the zero compensator *a* in the example.

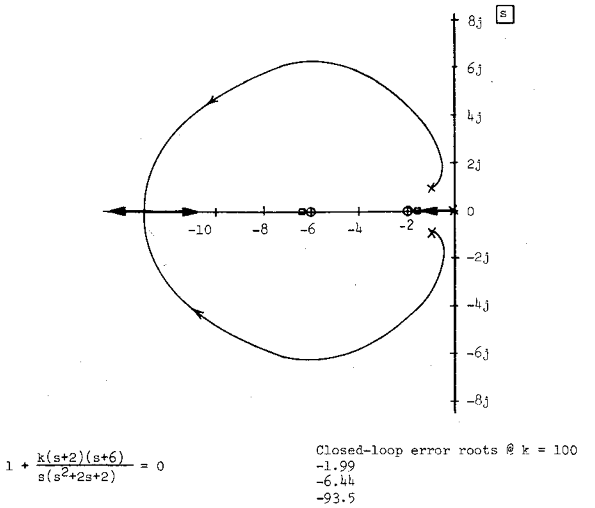

Fig. 5 Root locus for Eq. (29) using the linear criterion in Eq. (18).

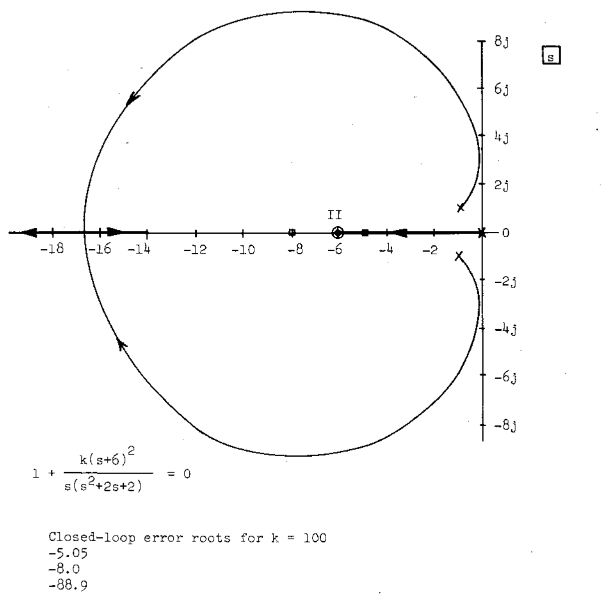

Fig. 6 Root locus for Eq. (29) using the nonlinear criterion in Eq. (28).

VI. Summary and Conclusions

Based on a study of the $\dot{V}$ term of a Lyapunov function, a new generalized stability criterion was derived which is applicable to a not too restrictive class of dynamical systems. The technique has been shown to be an extension of a traditional Lyapunov stability criterion, and includes the traditional case as a special case of the more general nonlinear criterion.

By use of a linearized error criterion, a comparison of the nonlinear criterion with the traditional linear case was effected. An example was used to illustrate the improvement in design freedom and response characteristics resulting from the nonlinear technique.

Some of the benefits of employing the extended stability criterion include 1) availability of a greater choice of response characteristics of the adaptive error; 2) calculations are straightforward and merely involve an extension of previously developed design methods; and 3) asymptotic stability of the error is still guaranteed. The shortcoming of the derived technique is that knowledge of the range of values of certain inputs is required. However, in many practical cases, such information is readily available a priori.

Table 1 Constants used in adaptive system simulations

Model: $\dfrac{2}{s^2+2s+2}$	Plant: $\dfrac{b}{s^2+cs+d}$
Linear criterion	*Nonlinear criterion*
$\beta_{21}=3$	$\beta_{21}=3$
$\alpha_{21}=18$	$\alpha_{21}=18$
$\gamma_1=6$	$\gamma_1=6$
$\delta_1=1$	$\delta_1=1$
$q_{12}=2, q_{22}=1$	$q_{12}=6, q_{22}=1$
$k=K_2q_{22}=100$	$k=K_2q_{22}=100$

$d=1.9$, $cc=2$, $b=2$ (b initially aligned but adapted)
$U=5\mu(t)$, step input
$x_{1_m}(0)=5$, $x_{1_p}(0)=4.9$, $e(0)=.1$
Terms b, d both were adapted

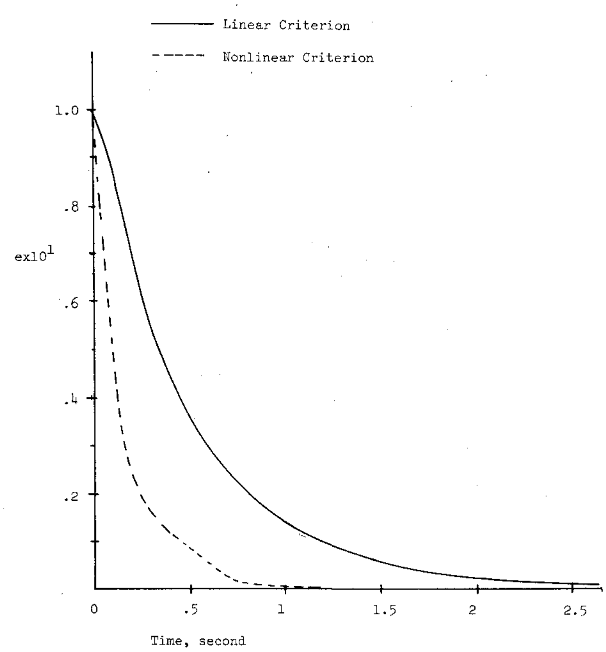

Fig. 7 Error responses comparsion results using the linear and nonlinear criteria with two terms adapting.

In conclusion, the nonlinear algebraic stability criterion can be of great value when some a priori knowledge of the range of the inputs is available and when dynamic response characteristics of the plant are important.

Appendix

A basic algorithm for determining solutions to Eq. (28), given W and Ω, is shown here. Given the direct matrix product formulation[34] in Ref. 35, as discussed in Ref. 36, and using a modified Newton-Raphson iterative method[37,38] for solution of simultaneous equations, a simple iterative technique can be derived. The form of Eq. (18) can be revised to

$$A\ x=b \tag{A1}$$

where

$$A=\frac{n(n+1)}{2}\ x\ \frac{n(n+1)}{2} \text{ matrix of } a_{ij}^m \text{ elements}$$

$$x^T=(q_{11}\ q_{12}\cdots q_{1n}\ q_{22}\cdots q_{nn})$$

$$b^T=(c_{11}\ c_{12}\cdots c_{1n}\ c_{22}\cdots c_{nn})$$

and similarly, Eq. (28) can be written as

$$Ax-\Omega \begin{bmatrix} q_{1n}^2 \\ 2q_{1n}q_{2n} \\ \cdot \\ \cdot \\ \cdot \\ 2q_{1n}q_{nn} \\ \cdot \\ \cdot \\ \cdot \\ q_{nn}^2 \end{bmatrix} = (w_{11}\ w_{12}\ \cdots\ w_{1n} w_{22}\ \cdots\ w_{nn})^T \quad \text{(A2)}$$

Equation (A2) represents a system of $j=[(n)(n+1)]/2$ equations in j variables, which can be written in the form

$$\begin{aligned} y_1 &= f_1(x_1,\ x_2,\ldots x_j) \\ y_2 &= f_2(x_1,\ x_2,\ldots x_j) \\ &\vdots \\ y_j &= f_j(x_1,\ x_2,\ldots x_j) \end{aligned} \quad \text{(A3)}$$

If $x_1^*,\ x_2^*,\ldots x_j^*$ represent the solution, then $y_\ell=0,\ \ell=1,2,\ldots j$. Expanding in a Taylor series, and neglecting higher-order-terms than first order, Eq. (A3) is rearranged as

$$\begin{aligned} \Delta f_1 &= y_1 - f_1(\hat{x}_1,\ \hat{x}_2,\ldots \hat{x}_j) \\ &= \frac{\partial f_1}{\partial x_1}\Big|_{\hat{x}}\Delta x_1 + \frac{\partial f_1}{\partial x_2}\Big|_{\hat{x}}\Delta x_2 + \ldots + \frac{\partial f_1}{\partial x_j}\Big|_{\hat{x}}\Delta x_j \\ \Delta f_2 &= y_2 - f_2(\hat{x}_1,\ \hat{x}_2,\ldots \hat{x}_j) \\ &= \frac{\partial f_2}{\partial x_1}\Big|_{\hat{x}}\Delta x_1 + \frac{\partial f_2}{\partial x_2}\Big|_{\hat{x}}\Delta x_2 + \ldots + \frac{\partial f_2}{\partial x_j}\Big|_{\hat{x}}\Delta x_j \\ \Delta f_j &= y_j - f_j(\hat{x}_1,\ \hat{x}_2,\ldots \hat{x}_j) \\ &= \frac{\partial f_j}{\partial x_1}\Big|_{\hat{x}}\Delta x_1 + \frac{\partial f_j}{\partial x_2}\Big|_{\hat{x}}\Delta x_2 + \ldots + \frac{\partial f_j}{\partial x_j}\Big|_{\hat{x}}\Delta x_j \end{aligned} \quad \text{(A4)}$$

where $\hat{x}$ represents the expansion point (the present approximation to the solution) and the equality $\hat{x}=x^*$ is being sought. Equation (A4) then can be written as

$$\Delta f = J\Big|_{\hat{x}}\Delta x \quad \text{(A5)}$$

where

$$\Delta f = -f\Big|_{\hat{x}} \text{(will be zero at } \hat{x}=x^*)$$

$$\Delta x = x_{\text{new}} - x_{\text{old}}$$

$J|_{\hat{x}}$ is the Jacobian of Eq. (A3) evaluated at the expansion point, which can be used to solve for updated x values through

$$\Delta x = J^{-1}\Delta f \quad \text{(A6)}$$

Using Eqs. (A1-A6), an iterative algorithm then is suggested

Initalization

Obtain $x(0)$ by solving Eq. (A1), representing

$$\Omega = 0 \quad \text{(A7)}$$

$$\text{compute } \Delta f(0) = 0 - f\Big|_{x(0)} \quad \text{(A8)}$$

$$\text{compute } \Delta x(0) \text{ using Eq. (A6)} \quad \text{(A9)}$$

Algorithm

$$x(k+1) = x(k) + \Delta x(k) \quad \text{(A10)}$$

$$f(k+1) = \begin{bmatrix} f_1[x_1(k+1),\ x_2(k+1),\ \cdots,\ x_j(k+1)] \\ f_2[x_1(k+1),\ x_2(k+1),\ \cdots,\ x_j(k+1)] \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ f_j[x_1(k+1),\ x_2(k+1),\ \cdots,\ x_j(k+1)] \end{bmatrix} \quad \text{(A11)}$$

$$\Delta f(k+1) = -f(k+1) \quad \text{(A12)}$$

$$\text{compute } J(k+1) = J\Big|_{x(k+1)} \quad \text{(A13)}$$

$$\Delta x(k+1) = J^{-1}(k+1)\Delta f(k+1) \quad \text{(A14)}$$

Repeat Eqs. (A10-A14) until a solution is reached. Then check the resulting Q matrix to insure p.d.

For the example in the text,

$$x^T = (q_{11}\ q_{12}\ q_{22})$$

The solution of the form in Eqs. (46) and (48), with $\Omega=0$, is

$$x^T(0) = (265 \quad 96 \quad 48.5)$$

With $\Omega_{\min}=10$, Eq.(A3) becomes

$$\begin{aligned} y_1 &= 0 = -4q_{12} - \Omega q_{12}^2 + 384 \\ y_2 &= 0 = q_{11} - 2q_{12} - 2q_{22} - \Omega q_{12}q_{22} + 24 \\ y_3 &= 0 = 2q_{12} - 4q_{22} - \Omega q_{22}^2 + 2 \end{aligned}$$

Using Eq. (A5)

$$J = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\partial f_1/\partial x_1 = 0 \quad \partial f_1/\partial x_2 = -4-2\Omega x_2 \quad \partial f_1/\partial x_3 = 0$$

$$\partial f_2/\partial x_1 = 1 \quad \partial f_2/\partial x_2 = -2-\Omega x_3 \quad \partial f_2/\partial x_3 = -2-\Omega x_2$$

$$\partial f_3/\partial x_1 = 0 \quad \partial f_3/\partial x_2 = 2 \quad \partial f_3/\partial x_3 = -4-2\Omega x_3$$

$$\Delta f(0)^T = (92160 \quad 46560 \quad 23522.5)$$

$$\Delta x(0) = \begin{bmatrix} 0 & -1924 & 0 \\ 1 & -487 & -962 \\ 0 & 2 & -974 \end{bmatrix}^{-1} \begin{bmatrix} 92160 \\ 46560 \\ 23522.5 \end{bmatrix}$$

$$\Delta x(0) = \begin{bmatrix} -.25414 & 1 & -.9876 \\ -.0005197 & 0 & 0 \\ -.0000010672 & 0 & -.001026 \end{bmatrix} \begin{bmatrix} 92160 \\ 46560 \\ 23522.5 \end{bmatrix}$$

$$\Delta x^T(0) = (-94.23 \quad -47.89 \quad -24.25)$$

$$x^T(1) = (170.77 \quad 48.11 \quad 24.25)$$

Repeated application of this method then yields a solution to the desired accuracy of Δf.

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